Rotation Matrices and Quaternion

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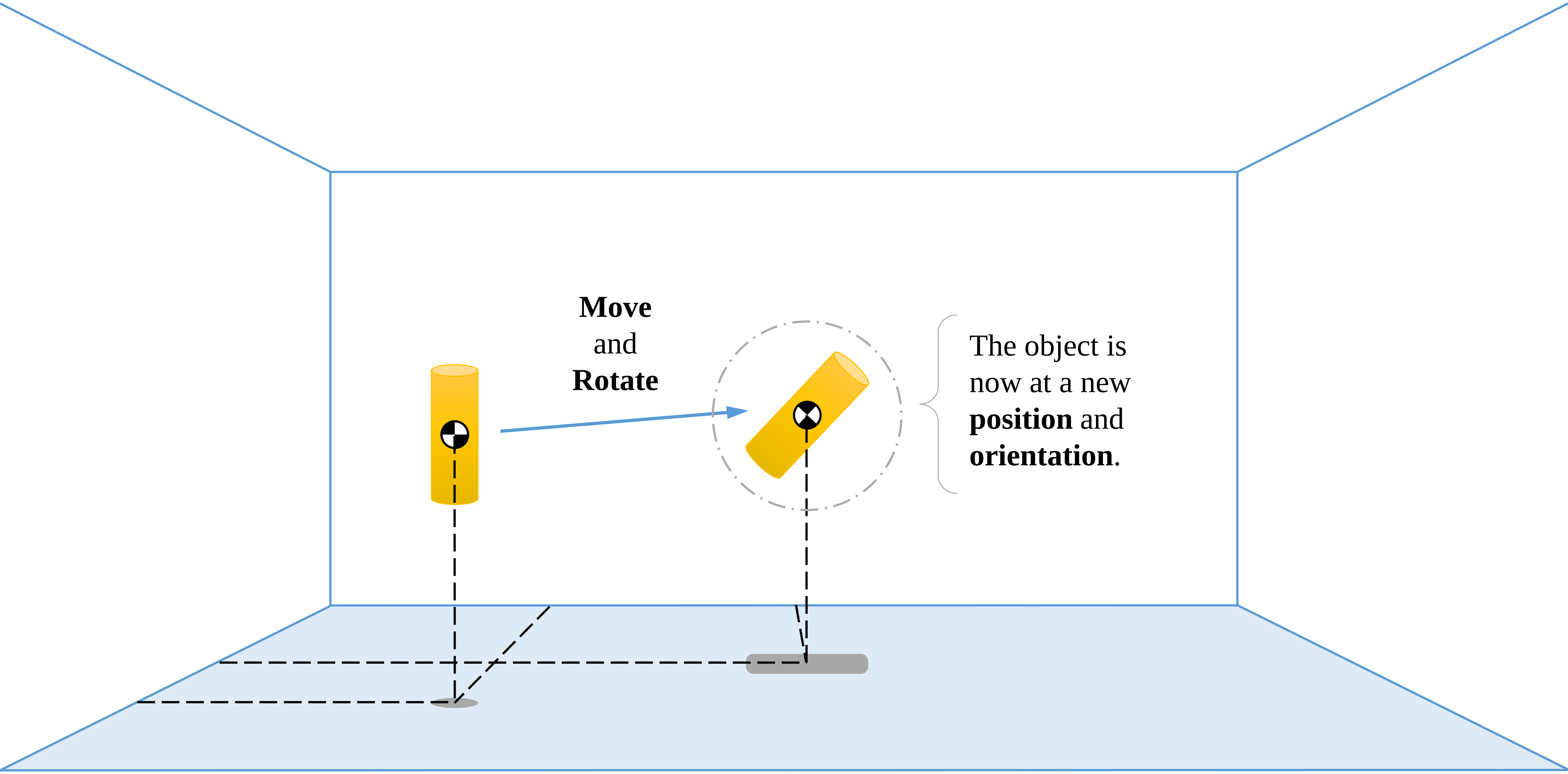
1/2

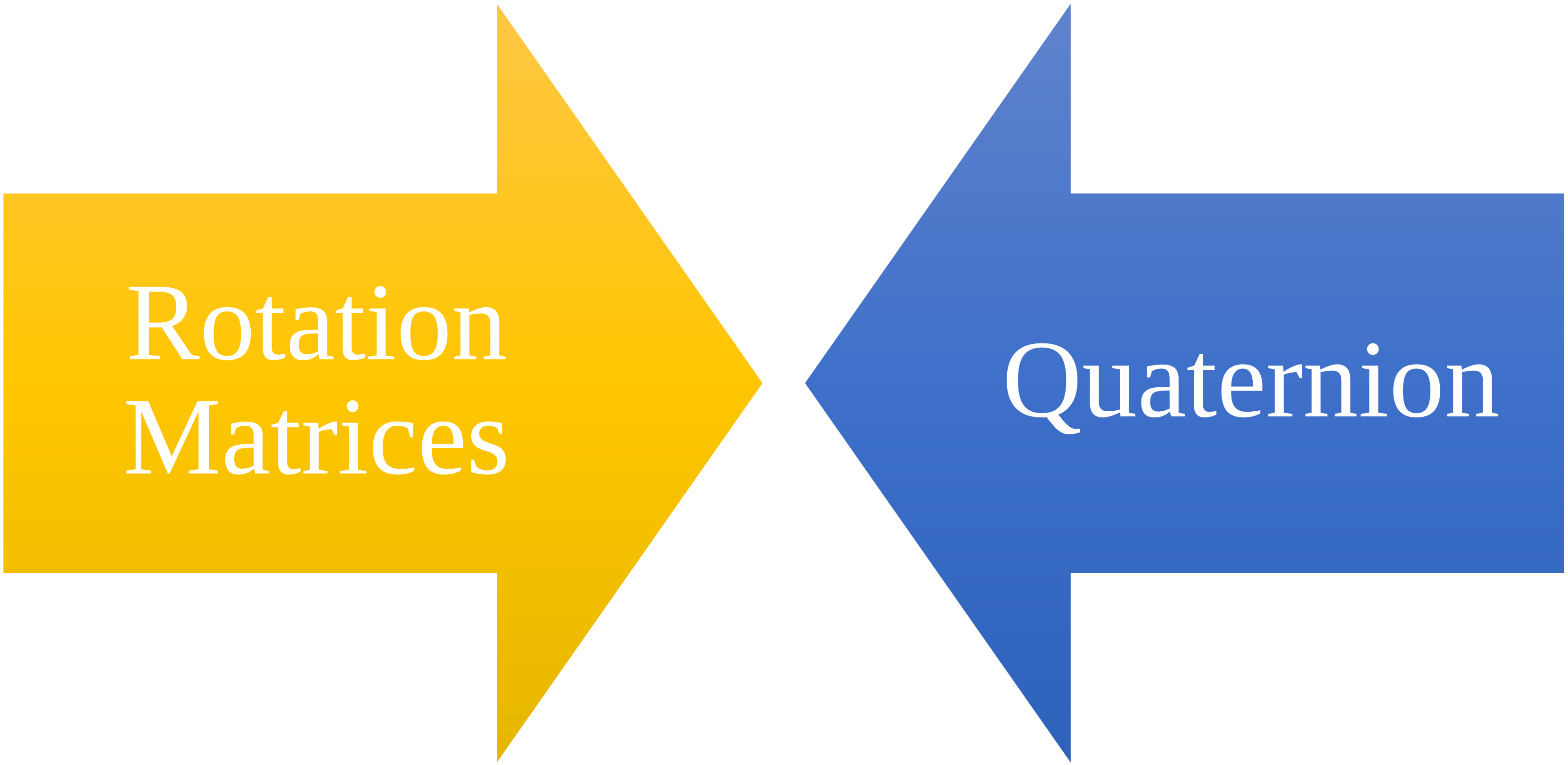
04/20/2016

lecture

Orientation and Rotation



Orientation and Rotation



Preliminaries and notations

- Unit vectors:

$$\hat{\imath} = [1 \quad 0 \quad 0]^T \quad ; \quad \hat{\jmath} = [0 \quad 1 \quad 0]^T \quad ; \quad \hat{k} = [0 \quad 0 \quad 1]^T$$

- Transpose:

$$[A^T]_{ij} = [A]_{ji}$$

- Inner product:

$$\langle x, y \rangle = x \cdot y = x^T y$$

- Notation simplifications:

$$s_\theta = \sin(\theta)$$

$$c_\theta = \cos(\theta)$$

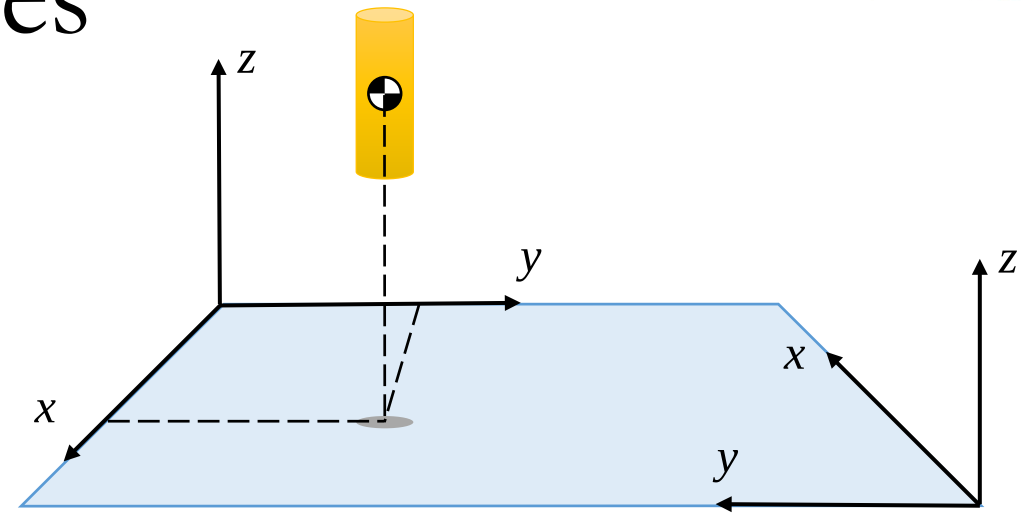
$$s_{12} = \sin(\theta_1 + \theta_2)$$

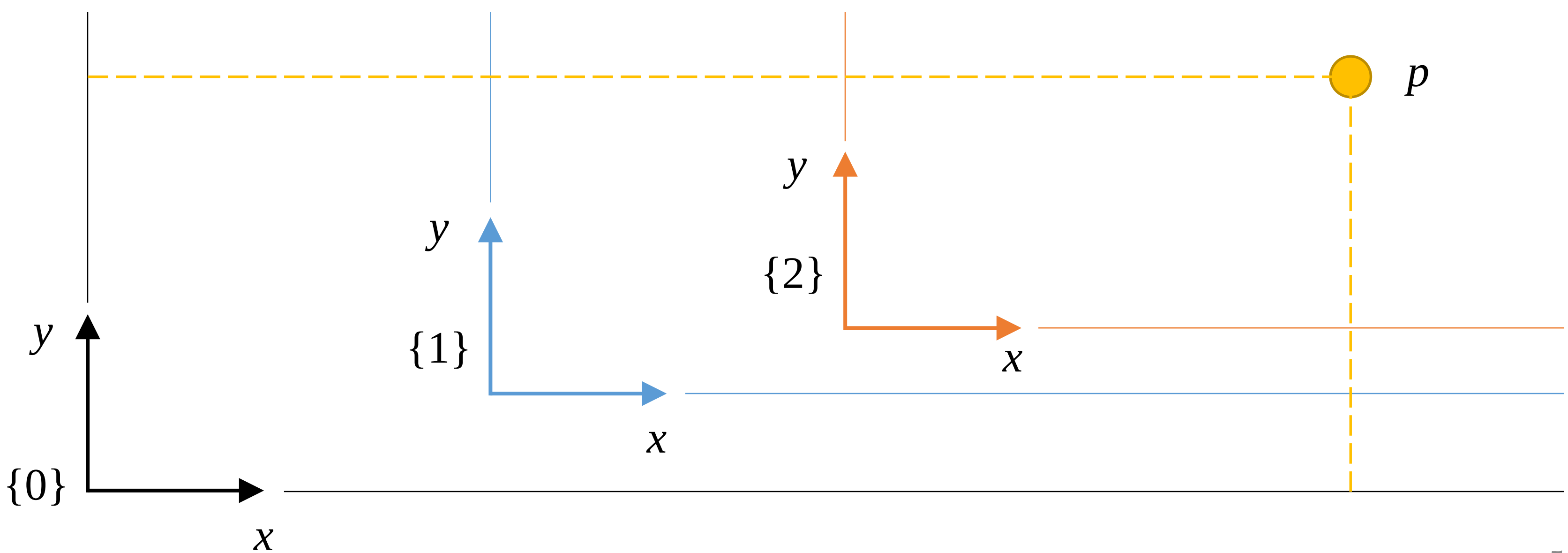
$$c_{12} = \cos(\theta_1 + \theta_2)$$

Reference Frames

- A point p can have different coordinates in different reference frames.

- Frames can be fixed in space or attached to moving objects (inertial or non-inertial frames)



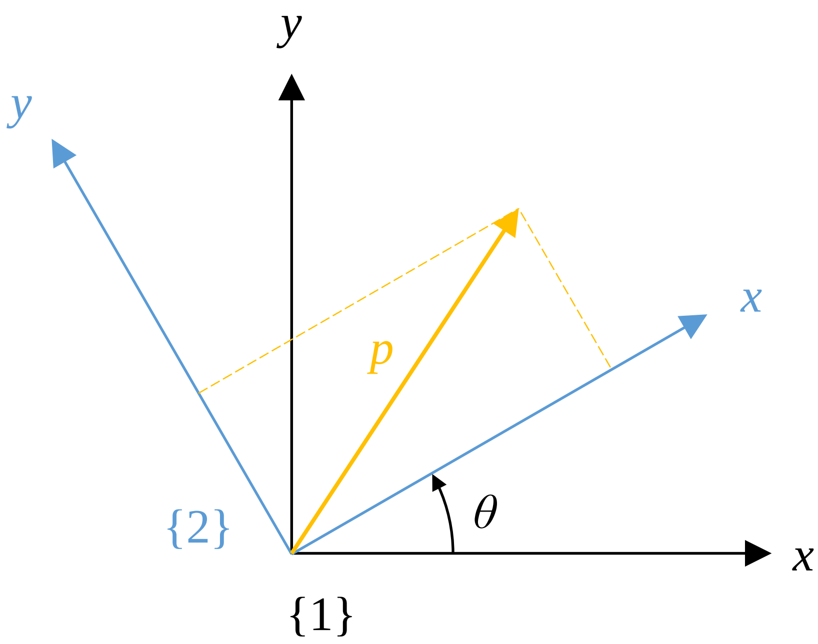


Rotation Matrix

- A 2D example

$$^{2}p = {}^{2}x_p \cdot {}^{2}\hat{\imath} + {}^{2}y_p \cdot {}^{2}\hat{\jmath}$$

$$^{1}x_p = {}^{2}x_p \langle {}^{2}\hat{\imath}, {}^{1}\hat{\imath} \rangle + {}^{2}y_p \langle {}^{2}\hat{\jmath}, {}^{1}\hat{\imath} \rangle$$
$$^{1}y_p = {}^{2}x_p \langle {}^{2}\hat{\imath}, {}^{1}\hat{\jmath} \rangle + {}^{2}y_p \langle {}^{2}\hat{\jmath}, {}^{1}\hat{\jmath} \rangle$$



$$\langle a, b \rangle = |a||b| \cos\theta$$
$$|\hat{\imath}| = |\hat{\jmath}| = 1$$

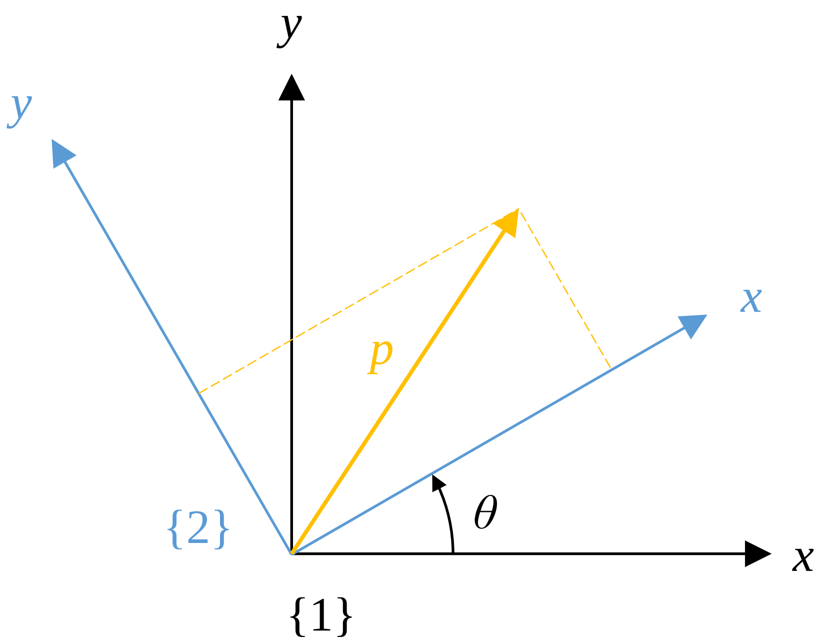
$$\begin{bmatrix} ^{1}x_p \\ ^{1}y_p \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \cos\left(\frac{\pi}{2} + \theta\right) \\ \cos\left(\frac{\pi}{2} - \theta\right) & \cos(\theta) \end{bmatrix} \begin{bmatrix} ^{2}x_p \\ ^{2}y_p \end{bmatrix}$$

$$\begin{bmatrix} ^{1}x_p \\ ^{1}y_p \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} ^{2}x_p \\ ^{2}y_p \end{bmatrix}$$

Rotation Matrix

- A 2D example

$$\begin{bmatrix} {}^{1}x_p \\ {}^{1}y_p \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} {}^{2}x_p \\ {}^{2}y_p \end{bmatrix}$$



$${}^{1}p = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} {}^{2}p$$

$${}^{1}p = {}^{1}_{2}R\ {}^{2}p$$

The R matrix is both and operator and a description!

Rotation Matrix (Direction Cosines)

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{\imath}_B & {}^{A}\hat{\jmath}_B & {}^{A}\hat{k}_B \end{bmatrix} $$

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{B}\hat{\imath} \cdot {}^{A}\hat{\imath} & {}^{B}\hat{\jmath} \cdot {}^{A}\hat{\imath} & {}^{B}\hat{k} \cdot {}^{A}\hat{\imath} \\ {}^{B}\hat{\imath} \cdot {}^{A}\hat{\jmath} & {}^{B}\hat{\jmath} \cdot {}^{A}\hat{\jmath} & {}^{B}\hat{k} \cdot {}^{A}\hat{\jmath} \\ {}^{B}\hat{\imath} \cdot {}^{A}\hat{k} & {}^{B}\hat{\jmath} \cdot {}^{A}\hat{k} & {}^{B}\hat{k} \cdot {}^{A}\hat{k} \end{bmatrix} $$

Standard Axis Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrix Identities

Claim: Rotation matrix from A to B is equal to the transpose of a rotation matrix from B to A

$${}^{A}_{B}R = \left({}^{B}_{A}R\right)^{T}$$

Proof:

$${}^{B}_{A}R = \begin{bmatrix} {}^{A}\hat{\imath} \cdot {}^{B}\hat{\imath} & {}^{A}\hat{\jmath} \cdot {}^{B}\hat{\imath} & {}^{A}\hat{k} \cdot {}^{B}\hat{\imath} \\ {}^{A}\hat{\imath} \cdot {}^{B}\hat{\jmath} & {}^{A}\hat{\jmath} \cdot {}^{B}\hat{\jmath} & {}^{A}\hat{k} \cdot {}^{B}\hat{\jmath} \\ {}^{A}\hat{\imath} \cdot {}^{B}\hat{k} & {}^{A}\hat{\jmath} \cdot {}^{B}\hat{k} & {}^{A}\hat{k} \cdot {}^{B}\hat{k} \end{bmatrix}$$

$$\Rightarrow \left({}^{B}_{A}R\right)^{T} = \begin{bmatrix} {}^{B}\hat{\imath} \cdot {}^{A}\hat{\imath} & {}^{B}\hat{\jmath} \cdot {}^{A}\hat{\imath} & {}^{B}\hat{k} \cdot {}^{A}\hat{\imath} \\ {}^{B}\hat{\imath} \cdot {}^{A}\hat{\jmath} & {}^{B}\hat{\jmath} \cdot {}^{A}\hat{\jmath} & {}^{B}\hat{k} \cdot {}^{A}\hat{\jmath} \\ {}^{B}\hat{\imath} \cdot {}^{A}\hat{k} & {}^{B}\hat{\jmath} \cdot {}^{A}\hat{k} & {}^{B}\hat{k} \cdot {}^{A}\hat{k} \end{bmatrix} = {}^{A}_{B}R$$

Rotation Matrix Identities

Claim: The transpose of a rotation matrix is equal to its inverse

$$ {}^{A}_{B}R = ({}^{B}_{A}R)^{-1} = ({}^{B}_{A}R)^{T} $$

Proof:

$$ {}^{A}_{B}R^{T}\ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{\imath}_{B}^{T} \\ {}^{A}\hat{\jmath}_{B}^{T} \\ {}^{A}\hat{k}_{B}^{T} \end{bmatrix} \begin{bmatrix} {}^{A}\hat{\imath}_{B} & {}^{A}\hat{\jmath}_{B} & {}^{A}\hat{k}_{B} \end{bmatrix} = I_3 $$

where I_3 is the 3×3 identity matrix. Hence,

$$ ({}^{B}_{A}R)^{-1} = ({}^{B}_{A}R)^{T} $$

Rotation Matrix Identities

Claim: The determinant of a rotation matrix is 1

$$\det(R) = 1$$

Proof:

$$R^T R = I \xRightarrow{\det(A)=\det(A^T)} (\det(R))^2 = \det(I) = \pm 1$$

Rotation matrix preserves dimensions, hence: $\|Rv\| = \|v\|$

$$\|Rv\| \leq \lambda_{max}(R)\|v\| \Rightarrow \lambda_{max}(R) = +1$$

$$\lambda_{min}(R)\|v\|_2^2 \leq \langle Rv, v\rangle \Rightarrow \lambda_{min}(R) v^T v \leq v^T R^T v$$

Using length perversity of R implies: $v^T v = \langle Rv, v\rangle$; thus: $\lambda_{min}(R) = 1$

Implies that all the eigenvalues of R are 1, consequently: $\det(R) = \lambda_1 \lambda_2 \lambda_3 = +1$

Rotation Matrix Identities

Claim: Multiplication of two rotation matrices is another rotation matrix

$$R = R_1 R_2$$

Proof:

Using $R^T = R^{-1}$ and consequently $R^T R = I$:

$$(R_1 R_2)^T (R_1 R_2) = R_2^T (R_1^T R_1) R_2 = I$$

Also:

$$\det(R_1 R_2) = \det(R_1) \det(R_2) = +1$$

Rotation Matrix Identities

$$\det(R) = 1$$

$$_B^A R = (_A^B R)^T$$

$$R^T = R^{-1}$$

$$R_3 = R_1 R_2$$

$$R_1 R_2 \neq R_2 R_1 \quad \forall n > 2$$

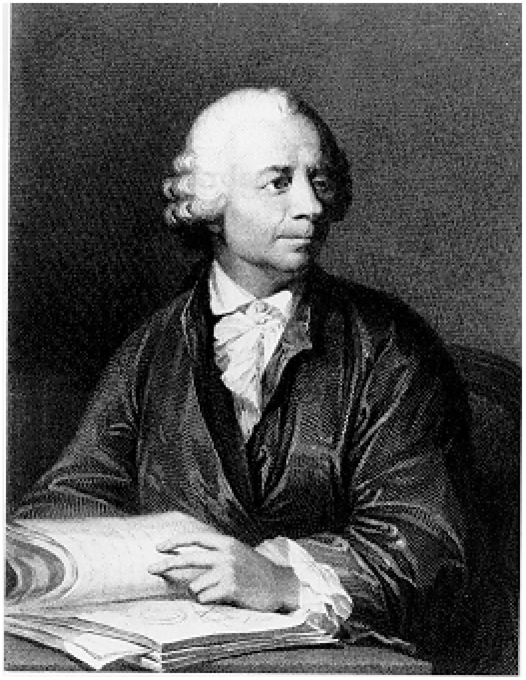
n is space dimension

Intrinsic vs. Extrinsic Rotations

Intrinsic rotations

Extrinsic rotations

Euler Angles

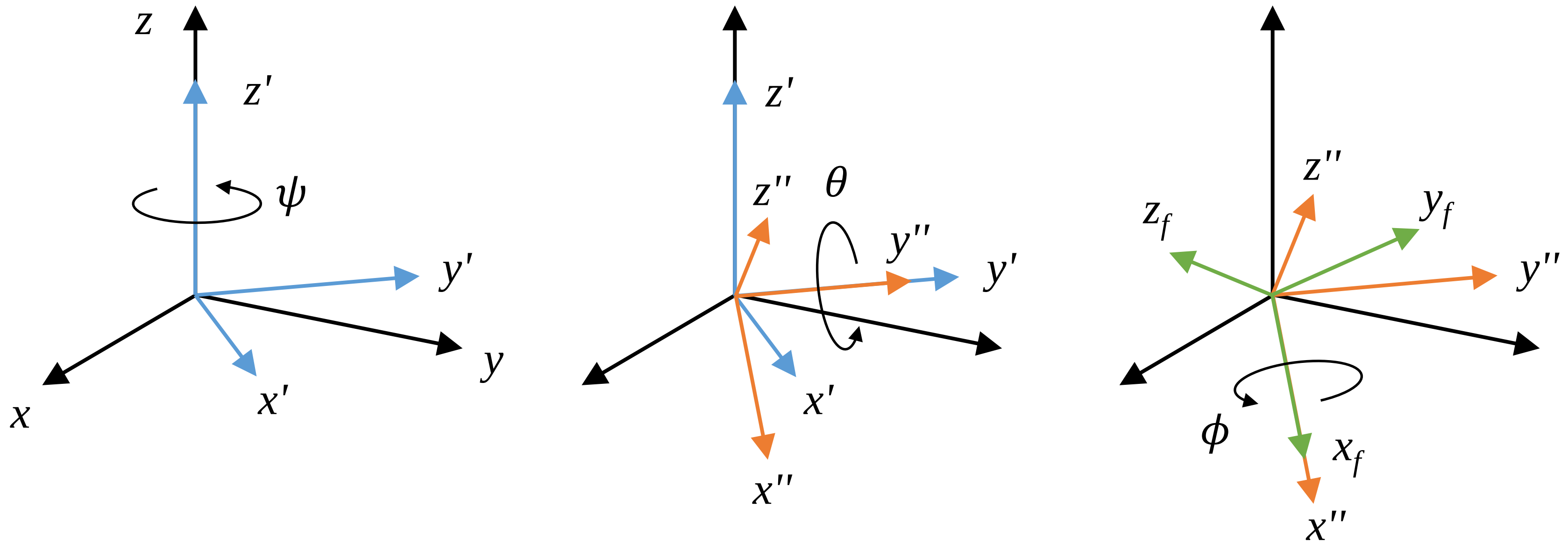


Leonhard Euler

April 15, 1707, Basel, Switzerland
September 18, 1783, Saint Petersburg, Russia

Intrinsic rotations (Euler)

- Rotations about "new" axes

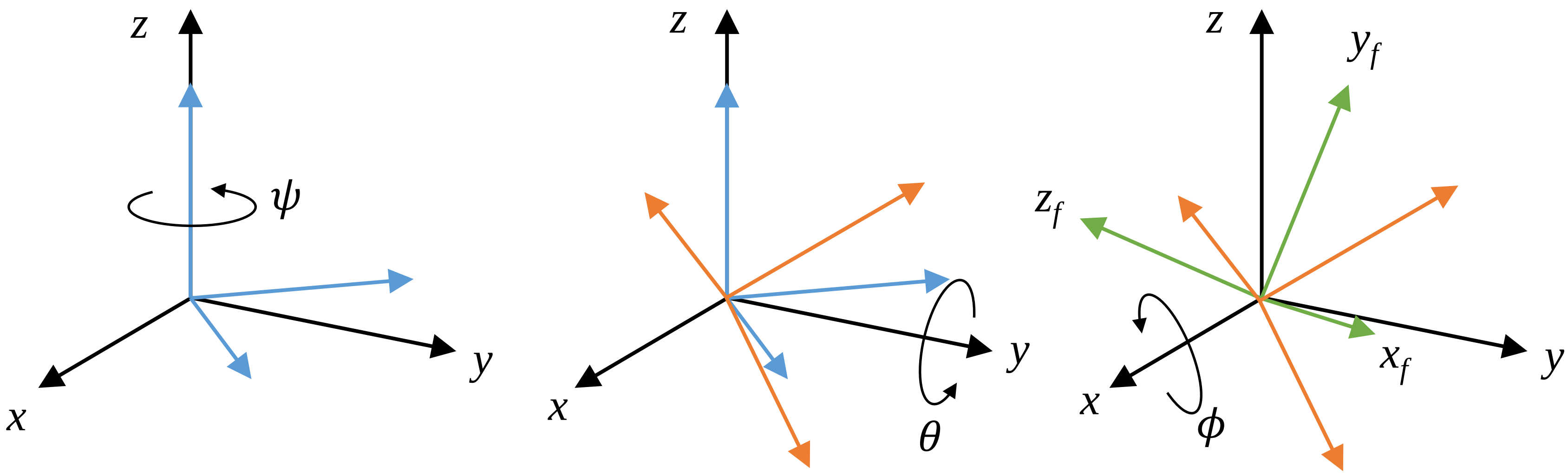


- For intrinsic rotations, rotation matrices are post-multiplied:

$$R = R_z(\psi)R_y(\theta)R_x(\phi)$$

Extrinsic rotations

- Rotations about “fixed” axes



- For extrinsic rotations, rotation matrices are pre-multiplied:

$$R = R_x(\phi)R_y(\theta)R_z(\psi)$$

More on Rotation Matrix

$R^{-1} = R^T \Rightarrow R$ is an orthogonal matrix, thus for:

$$R = [\hat{x} \quad \hat{y} \quad \hat{z}]$$

The following statements hold:

$$\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0$$

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$$

This implies only 3 of the 9 components of R are independent!

Consequently, any orientation is achievable through 3 successive rotations about linearly independent axes.

Euler Angles

- Proper Euler angles:

  z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y

- Tait-Bryan angles:

  x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z

z-y-x Euler Angles

$$ {}^{A}_{B}R_{zyx} = R_z(\psi)R_y(\theta)R_x(\phi) $$

$$ {}^{A}_{B}R_{zyx} = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} $$

$$ {}^{A}_{B}R_{zyx} = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} $$

z-y-x Euler Angles

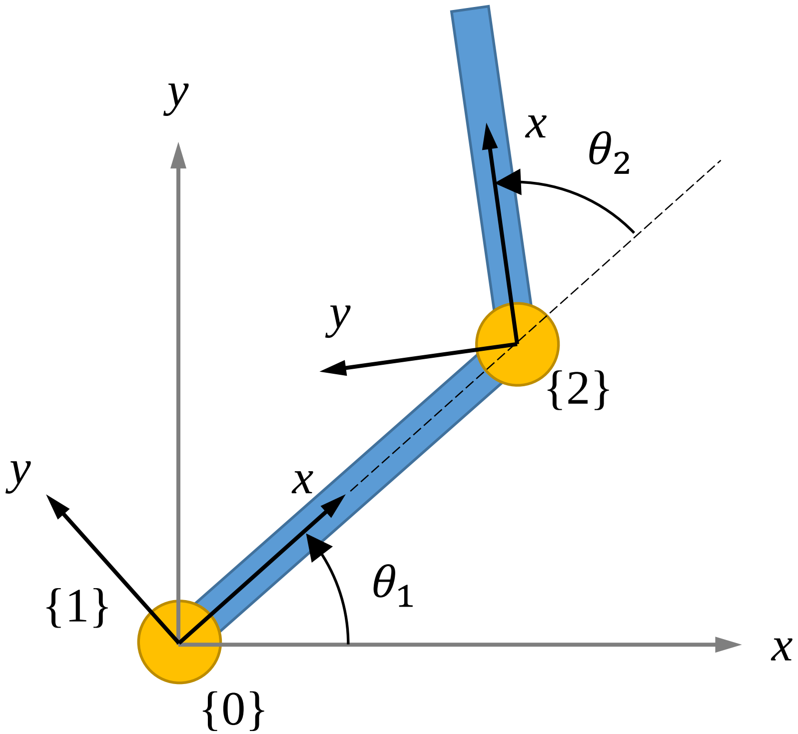
$$ {}^{A}_{B}R_{zyx} = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} $$

$$ {}^{A}_{B}R = [r_{ij}] $$

if $c_\theta \neq 0$	if $\theta = 90°$	if $\theta = -90°$
$\theta = \text{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$	$\psi = 0$	$\psi = 0$
$\psi = \text{atan2}\left(\frac{r_{21}}{c_\theta}, \frac{r_{11}}{c_\theta}\right)$	$\phi = \text{atan2}(r_{12}, r_{22})$	$\phi = -\text{atan2}(r_{12}, r_{22})$
$\phi = \text{atan2}\left(\frac{r_{32}}{c_\theta}, \frac{r_{33}}{c_\theta}\right)$		

Rotations in Constrained Objects

- Exercise: find ${}^0_2R$



$$ {}^0_1R = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 \\ s_{\theta_1} & c_{\theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

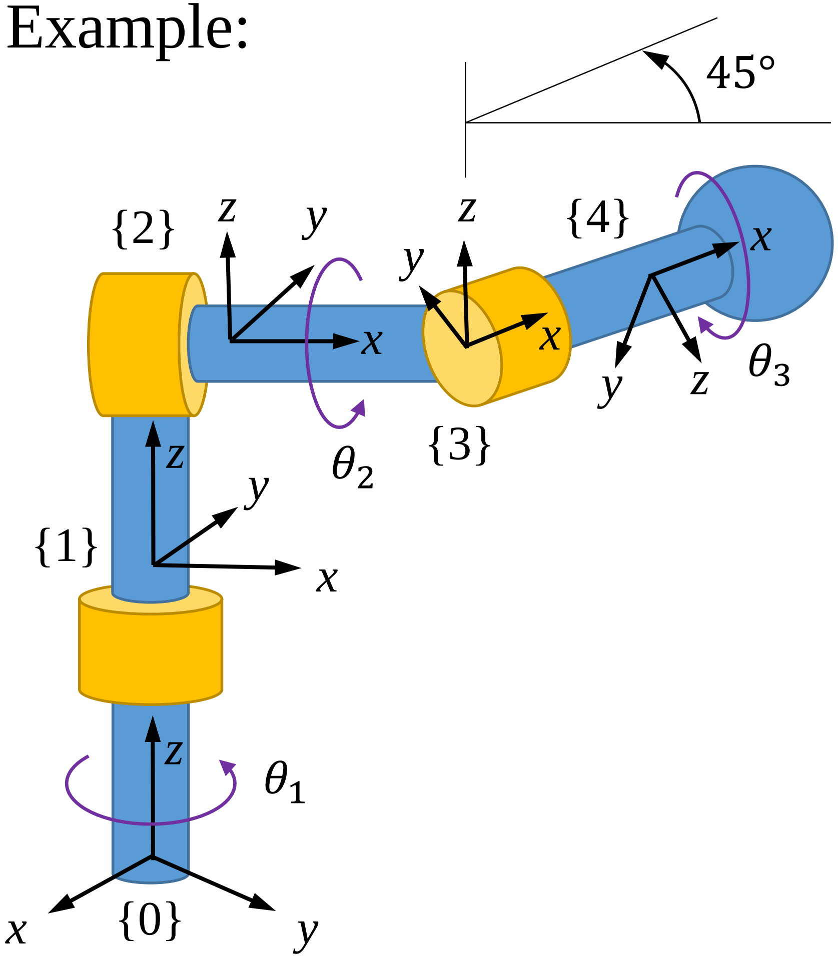
$$ {}^1_2R = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 \\ s_{\theta_2} & c_{\theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_2R = {}^0_1R\,{}^1_2R $$

$$ {}^0_2R = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Rotations in Constrained Objects

Example:



$${}^0_1R = R_z(\theta_1)$$

$${}^1_2R = R_x(\theta_2)$$

$${}^2_3R = R_z(45°)$$

$${}^3_4R = R_x(\theta_3)$$

$${}^2_4R = {}^2_3R\,{}^3_4R$$

$${}^1_4R = {}^1_2R\,{}^2_4R = {}^1_2R\,{}^2_3R\,{}^3_4R$$

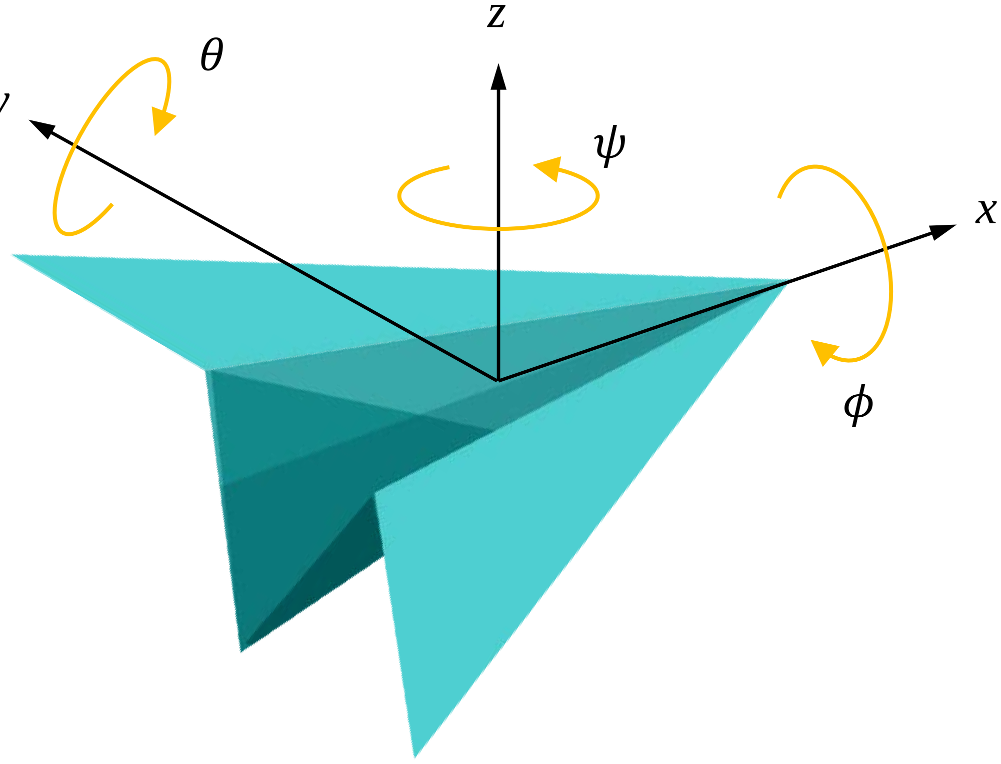
$${}^0_4R = {}^0_1R\,{}^1_4R = {}^0_1R\,{}^1_2R\,{}^2_3R\,{}^3_4R$$

Roll-Picth-Yaw

x-*y*-*z* or *z*-*y*-*x* Euler Angles

$$R_1 = R_x(\phi)R_y(\theta)R_z(\psi)$$

$$R_2 = R_x(\phi)R_y(\theta)R_z(\psi)$$



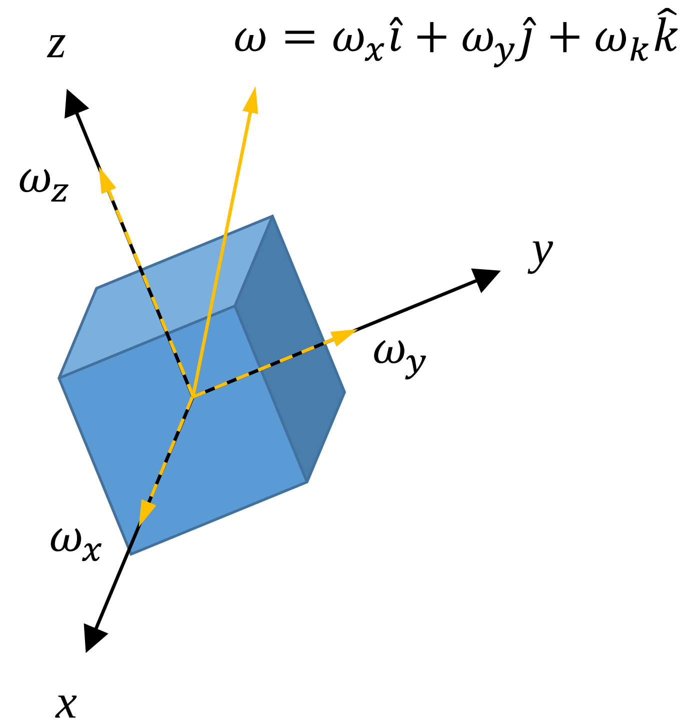
Rotations in free bodies

Question:

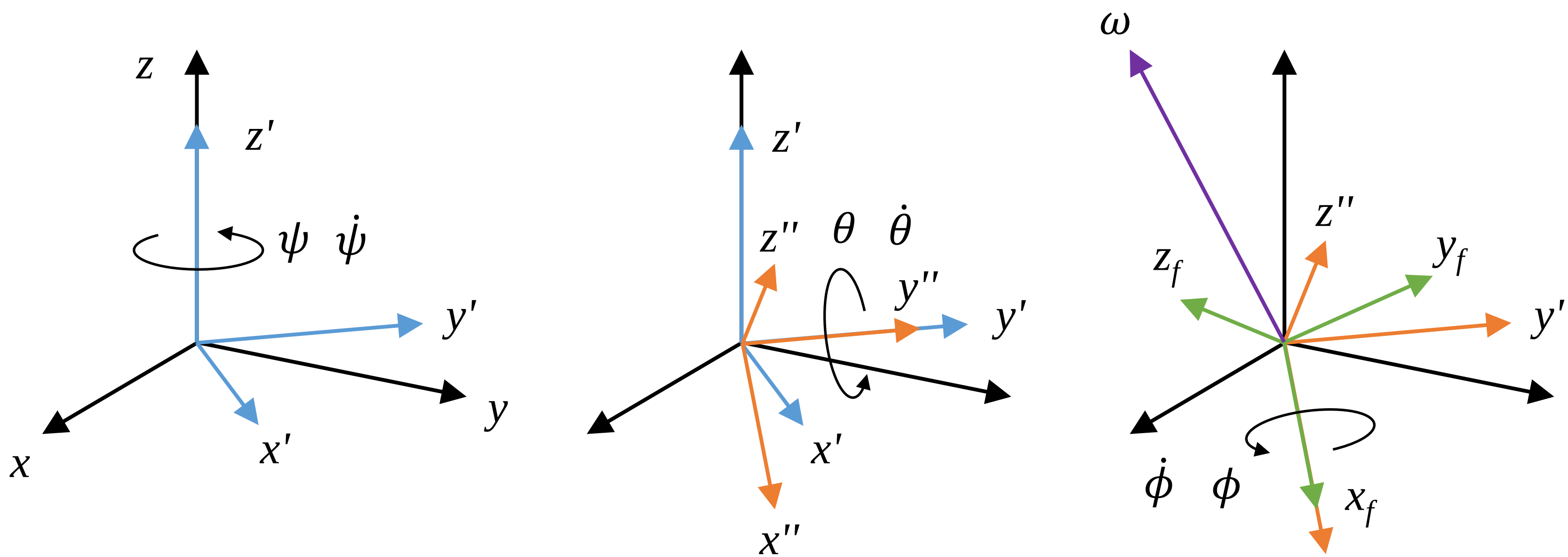
If we assume any type of Euler angles, how can we guarantee that rotations are preserved?

Answer:

We can not!



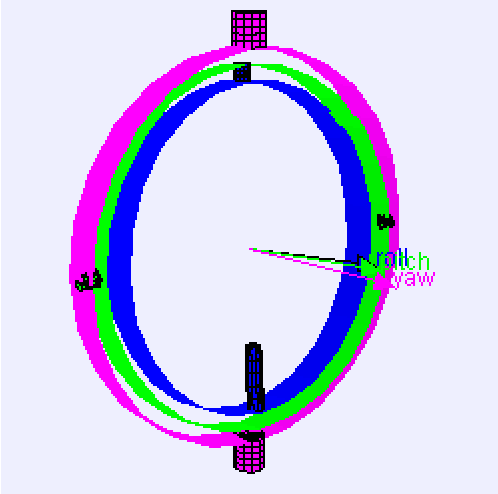
Rotations in free bodies



$$\omega = \dot{\phi}\hat{\imath} + [R_x(\phi)]^T \dot{\theta}\hat{\jmath} + \left[R_y(\theta)R_x(\phi)\right]^T \dot{\psi}\hat{k}$$

Singularity!

$$\omega = \dot{\phi}\hat{\imath} + [R_x(\phi)]^T\dot{\theta}\hat{\jmath} + \left[R_y(\theta)R_x(\phi)\right]^T\dot{\psi}\hat{k}$$



$$\omega_x = \dot{\phi} - \dot{\psi}\sin\theta$$

$$\omega_y = \dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi$$

$$\omega_z = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$

$$\dot{\phi} = \omega_x + \omega_y\sin\phi\tan\theta + \omega_z\cos\phi\tan\theta$$

$$\dot{\theta} = \omega_y\cos\phi - \omega_z\sin\phi$$

$$\dot{\psi} = \left(\omega_y\sin\phi + \omega_z\cos\phi\right)/\cos\theta$$

Singularities at: $\theta = \dfrac{(2k-1)\pi}{2}$

Time derivative of a Rotation Matrix

For angular velocity vector ω defined in body frame:

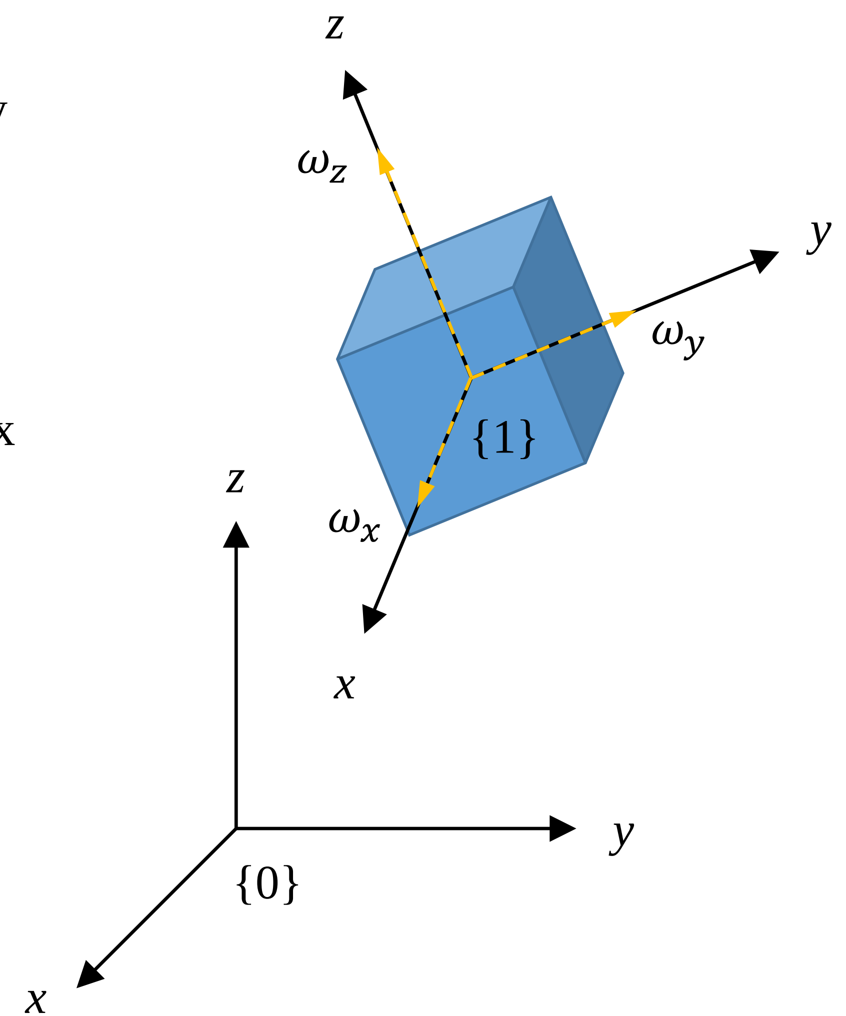
$$\omega = \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_k \hat{k}$$

And a rotation matrix ${}^0_1R$ that maps $\{1\}$ into $\{0\}$. The time derivative of the rotation matrix is defined as:

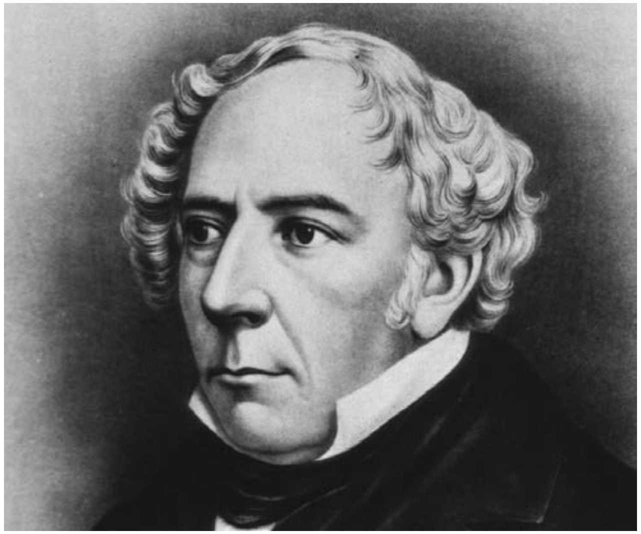
$$\frac{d}{dt}R(t) = S(\omega)R$$

where:

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$



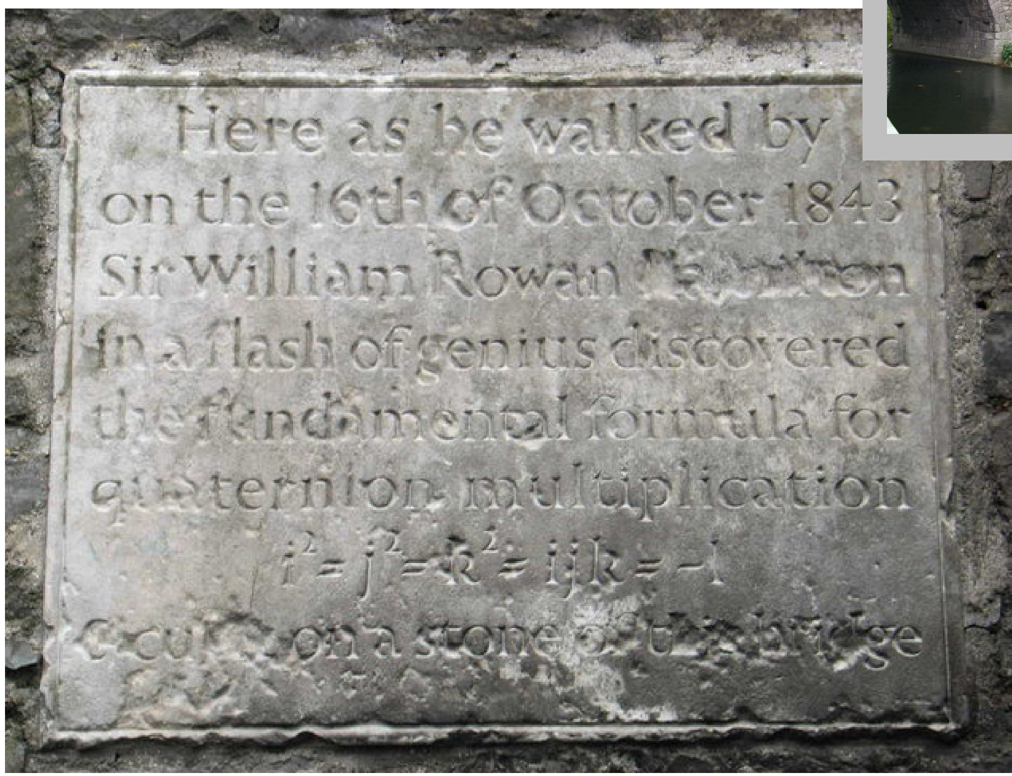
Quaternion



Sir William Rowan Hamilton

August 4, 1805, Dublin, Republic of Ireland
September 2, 1865, Dublin, Republic of Ireland

Quaternion



Broom Bridge
Dublin, Irland

$$i^2 + j^2 + k^2 = ijk = -1$$

End of Session 1 of 2

To be continued…

References:

A good reference on Rotation matrices and some of the underlying theories is presented in chapter 2 of:

- Craig, John J. “Introduction to robotics: mechanics and control.” Vol. 3. Upper Saddle River: Pearson Prentice Hall, 2005.

Proof of the time derivative of a rotation matrix:

- Hamano, Fumio. “Derivative of Rotation Matrix Direct Matrix Derivation of Well Known Formula.” arXiv preprint arXiv:1311.6010 (2013).